UNIT-3

BEHAVIOUR OF MONOTONIC SEQUENCES:

The following theorem gives the complete behaviour of monotonic sequences.

Theorem 3.21:

i. A monotonic increasing sequence which is bounded above converges to its *lub*.
ii. A monotonic increasing sequence which is not bounded above diverges to ∞.
iii. A monotonic decreasing sequence which is bounded below converges to its *glb*
iv. A monotonic decreasing sequence which is not bounded below diverges to $-\infty$.

Proof:

i. Let (a_n) be a monotonic increasing sequence which is bounded above.

Let k be the *lub* of the sequence

Then $a_n \leq k$ for all n. .... (1)

Now, let $\varepsilon > 0$ be given.

$\therefore$ $k-\varepsilon < k$ and hence $k-\varepsilon$ is not an upper bound of (a_n).

Hence, there exists a_m such that $a_m > k-\varepsilon$

Now, since (a_n) is monotonic increasing, $a_n \geq a_m$ for all $n \geq m$.

Hence $a_n > k-\varepsilon$ for all $n \geq m$........................ (2)

$\therefore K-\varepsilon < a_n \leq k$ for all $n \geq m$ (by (1) and (2))

$\therefore |a_{n-k}| < \varepsilon$ for all $n \geq m$.

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$\therefore (a_n) \to k.$

(ii) Let (a_n) be a monotonic increasing sequences which is not bounded above.

Let k>0 be any real number.

Since (a_n) is not bounded, there exists $m \in N$ such that $a_m > k$. Also $a_n \geq a_m$ for all $n \geq m$.

$\therefore a_n > k$ for all $n \geq m$.

$\therefore (a_n) \to \infty$.

(iii) Let a_n be a monotonic decreasing sequence which is bounded below.

Let l **be** the *glb* **of** the (a_n)............................ (1)

Than $a_n \geq k$ be given $l+\varepsilon > l$

Hence l-ε is not an lower bound of (a_n).

Hence there exists a_m such that $a_m < l + \varepsilon$............... (2)

Now, Since (a_n) is monotonic decreasing $a_m > a_n$ for all $n \geq m$.

$a_n < l + \varepsilon$

$l \leq a_n < l + \varepsilon$ (by(2)and(3))

$|a_n - l| < \varepsilon$ for all $n \geq m$.

$(a_n) \to l$

(iv) Let (a_n) be a monotonic decreasing sequence which is not bounded below.

Let k<0 be any real number.

Since (a_n) is not bounded below.

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There exists $m \in N$ such $a_m < k$.................... (1)

Also $a_n \leq a_m$ (by (1) and (2)) .... (2)

$A_n < k$ for all $n \geq m$

$(a_n) \to -\infty$

NOTE: The above theorem shows that a monotonic sequence either converges or diverges. Thus a *monotonic sequence cannot* be an oscillating sequence.

SOLVED PROBLEMS

PROBLEM: 1

Let $a_n = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\ldots+\frac{1}{n!}$. Show that $\lim_{n\to\infty} a_n$ exists and lies between 2 and 3.

Solution: Clearly (a_n) is a monotonic increasing sequence.

Also $a_n = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\ldots+\frac{1}{n!}$

$$\leq 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots+\frac{1}{n!}$$

$$=1+\left(1-\frac{1}{2^n}/1-\frac{1}{2}\right)$$

$$=1+\left(1-\frac{1}{2^n}\backslash\frac{2-1}{2}\right)$$

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

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$$=1+\left(1-\frac{1}{2^n}\backslash\frac{1}{2}\right)$$

$$=1+2\left(1-\frac{1}{2^n}\right) \qquad =1+2-\frac{2}{2^n}$$

$$=3-\frac{1}{2^{n-1}}<3 \qquad =a_n<3$$

$\therefore (a_n)$ is bounded above.

$\therefore \lim_{n\to\infty} a_n$ exists.

Also $2<a_n<3$ for all n.

$$\therefore 2\leq \lim_{n\to\infty} a_n \leq 3$$

Hence the result.

NOTE: The limit of the above sequence is denoted by e.

PROBLAM: 2

Show that the sequence $\left(1+\frac{1}{n}\right)^n$ converges.

SOLUTION:

$$a_n=1+1+\frac{n(n-1)}{2!}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}+\dots\dots\dots+\frac{1}{n^n}$$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\dots\dots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\dots\left(1-\frac{n-1}{n}\right).$$

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$=1+1+\frac{1}{2!}+\frac{1}{3!}+\dots\dots\dots+\frac{1}{n!}$.

$a_n < 3 \qquad 2 \le \lim_{a} n \le 3$

$\therefore (a_n)$ is bounded above.

Also,

$a_{n+1}=1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+}\right)+\dots\dots+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\dots\dots\left(1-\frac{n}{n+1}\right)$

$>1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\dots\dots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-1}{n}\right)$.

$\therefore a_{n+1} > a_n$

$\therefore (a_n)$ is monotonic increasing.

$\therefore (a_n)$ is a convergent sequence.

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1. show that $\lim_{n\to\infty}(1+\frac{1}{n})^n = \lim_{n\to\infty}(1+\frac{1}{1!}+\cdots+\frac{1}{n!}) = e$

solution

let $a_n = (1+\frac{1}{n})^n$ and $b_n = 1+\frac{1}{1!}+\ldots\ldots+\frac{1}{n!}$

then $a_n < b_n$ for all n

$\therefore \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$ ………… ①

Now

Let $m > n$

$a_m = (1+\frac{1}{m})^m$

$= 1+1+\frac{1}{2!}(1-\frac{1}{m})+\frac{1}{3!}(1-\frac{1}{m})(1-\frac{2}{m})+\ldots+\frac{1}{n!}(1-\frac{1}{m})\ldots$

$(1-\frac{n-1}{m})+\ldots\ldots+\frac{1}{m!}(1-\frac{1}{m})\ldots(1-\frac{m-1}{m})$

$> 1+1+\frac{1}{2!}(1-\frac{1}{m})+\cdots\cdots+\frac{1}{n!}(1-\frac{1}{m})\cdots\cdots(1-\frac{n-1}{m})$

Fixing n and taking lim as $m\to\infty$ *we* get

$\lim_{m\to\infty} a_m \ge 1+1+\frac{1}{2!}+\cdots\cdots\frac{1}{n!} = b_n$

Now taking limit as $n\to\infty$ we get

$\lim_{n\to\infty} a_n \ge \lim_{n\to\infty} b$ ……………… ②

$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = e$ (*b* ① *and* ②)

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2. let $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ show that (a_n) converges

Solution

$$a_{n+1} - a_n$$

$$= \left(\frac{1}{n+2} + \cdots + \frac{1}{2n+1} \right) - \left(\frac{1}{n+1} + \cdots + \frac{1}{n+n} \right)$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} - \frac{1}{2n+2} > 0 \text{ for all n}$$

$\therefore a_{n+1} > a_n$ for all n

$\therefore (a_n)$ is a monotonic increasing sequence

Also

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$

$$< \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = 1 \text{ for all n}$$

$\therefore (a_n)$ **is** bounded above

$\therefore (a_n)$ is converges

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3 .let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{n}$ **show that** (a_n) diverges to ∞

Solution

Clearly (a_n) is a monotonic increasing sequence

Now let $m = 2^n - 1$

$$a_n = 1 + \frac{1}{2} + \cdots\cdots + \frac{1}{2^n - 1}$$

$$= \mathbf{1} + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \cdots\cdots + \left(\frac{1}{2^n - 1} + \cdots\cdots \frac{1}{2^n - 1}\right)$$

$$> 1 + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots\cdots + \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right)$$

$$= 1 + (n - 1)\frac{1}{2}$$

$$= \frac{1}{2}(n + 1)$$

$$\therefore (a_n) > \frac{1}{2}(n + 1)$$

$\therefore (a_n)$ Is not bounded above

Hence $(a_n) \to \infty$

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4. Prove that ($n!/n^n$) converges .

Solution:

Let, $a_n = \frac{n!}{n^n}$

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

Then, $\frac{a_n}{a_{n+1}} = \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!}$

$$\frac{a_n}{a_{(n+1)}} = \frac{n!}{n^n} \times \frac{(n+1)^n.(n+1)}{n!.(n+1)}$$

$$\frac{a_n}{a_{n+1}} = \left(\frac{n+1}{n}\right)^n > 1$$

$\therefore$ $a_n > a_{n+1}$ for all $n \in N$

$\therefore$ (a_n) is a monotonic decreasing sequence.

Also, $a_n > 0$ for all $n \in N$

(a_n) Is a bounded below .

If (a_n) converges.

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1. Discuss the behavior of the geometric sequence (r^n)

SOLUTION :

Case(i):

Let $r = 0$

Then (r^n) reduces to the constant sequence 0,0.......and hence converges to 0

Case(ii):

Let $r = 1$

In this case (r^n) reduces to the constant sequence 1,1,1,........ and hence converges to 1

Case (iii):

Let $0 < r < 1$

In the case ,(r^n) is a montonic decreasing sequence and $(r^n) > 0$ for all $n \in N$.

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(r^n) is monotonic decreasing and bounded below and hence (r^n) converges.

Let $(r^n) \to l$

Since $r^n > 0$ for all n, $l > 0$. ....(1)

We claim that $l = 0$.

Let $\varepsilon > 0$ be given. Since $(r^n) \to l$, there exists $m \in N$ such that $l < r^n < l + \varepsilon$ for all $n \geq m$.

Fix $n > m$. Then $l < r^{n+1}$ ....(2)

Also $r^{n+1} = r.r^n < r(l+\varepsilon)$. ....(3)

$L < r(l+\varepsilon)$ (by 2 and 3).

$L < (r/1-r)\, \varepsilon$.

Since this is true every $\varepsilon > 0$, we get $l \leq 0$. ....(4)

$L = 0$ (by 1 and 4).

Case (iv):

Let $-1 < r < 0$.

Then $r^n = (-1)^n |r|^n$ where $0 < |r| < 1$.

By case (iii) $(|r|^n) \to 0$.

Also $((-1)^n)$ is a bounded sequence.

$((-1)^n |r|^n)$ converges to 0 (by problem 4 of 3.6)

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$(r^n) \to 0$.

Case (v)

Let $r = -1$.

In this case (r^n) reduces to $-1, 1, -1, \ldots\ldots$ which oscillates finitely.

Case(vi):

Let $r > 1$.

Then $0 < 1/r < 1$ and hence $(1/r^n) \to 0$ (case(iii))

$(r^n) \to \infty$ (by theorem 3.5)

Case(vii):

Let $r < -1$.

Then the terms of the sequence (r^n) are alternatively positive and negative . Also $|r| > 1$ and hence by case (vi) $(|r|^n)$ is unbounded.

(r^n) oscillates infinitely.

Thus ,

(i) If $r=1$, then $(r^n) \to 1$

(ii) If $r=0$, then $(r^n) \to 0$

(iii) If $0<r<1$, then $(r^n) \to 0$

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(iv) If $-1<r<0$, then $(r^n) \to 0$

(v) If $r=-1$, then oscilating finitely

(vi) If $r>1$ then $(r^n) \to \infty$

(vii) If $r<-1$ then (r^n) oscilating infinitely

2. Show that if $|r|<1$ then $(n r^n) \to 0$.

SOLUTION:

The result is trivial if $r=0$

Let $0<|r|<1$.

Then $|r| = 1/1+p$, $p>0$.

$|r|^n = 1/(1+p)^n$

$= 1/1+np+\{n(n-1)/1.2\} p^2+........$

$< 2/n(n-1)p^2$

$|n r^n| < 2/(n-1)p^2$

Now, let $\varepsilon > 0$ be given.

Then $2/(n-1)p^2 < \varepsilon$ provided $n > 1 + 2/p^2 \varepsilon$

$|n r^n| < \varepsilon$ if $n > 1 + 2/p^2 \varepsilon$.

$\lim_{n \to \infty} n r^n = 0$

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3. Show that $\lim_{n\to\infty} \log n/n^p = 0$ if $p>0$.

SOLUTION :

We have $e^p > 1$ (since $e > 1$)

$\therefore 1/e^p < 1.$

$\therefore [n/(e^p)^n] \to 0$ (by problem 8)

$\therefore$ Given $\varepsilon > 0$, there exists a natural number m such that $n/(e^p)^n < \varepsilon/e^p$ for all $n \geq m$. ....(1)

Now , let g be the positive integer such that

$g \leq \log n < (g+1)$ ....(2)

$\therefore \log n/ n^p < g+1/n^p$.

$\leq g+1/(e^g)^p$ (since $e^g \leq n$ by(2))

$= e^p\,(g+1)/e^{p(g+1)}$

$< e^p\,[\varepsilon/e^p]$ provided $g+1 \geq m$ (using 1)

$\therefore \log n/n^p < \varepsilon$ provided $g+1 \geq m$.

Now, if $n \geq e^m$, then $\log n \geq m$.

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But $g+1 > \log n$ (by(2))

$\therefore n \geq e^m \Rightarrow g + 1 \geq$

$\therefore \log n/n^p < \varepsilon$ provided $n \geq e^m$

$\therefore \lim_{n\to\infty} \log n/ n^p = 0.$

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Problem

Let (a_n) and (b_n) be two sequence of positive terms such that $a_{n+1} = \frac{1}{2}(a_n + b_n)$ and $b_{n+1} = \sqrt{(a_n b_n)}$. Prove that (a_n) and (b_n) convergence to the same limit.

Solution

By hypothesis, a_{n+1} and b_{n+1} are respective the A.M. and G.M. between a_n and b_n.

Also we know that A.M. $\geq$ G.M.

Hence $a_{n+1} \geq b_{n+1}$ ……………………… (1)

Moreover the A.M. and G.M of two numbers lie between the two numbers.

$\therefore\ a_n \geq a_{n+1} \geq b_n$ for all $n \in N$. ………………. (2)

and $a_n \geq b_{n+1} \geq b_n$ for all $n \in N$ ………………. (3)

$\therefore\ a_n \geq a_{n+1} \geq b_{n+1} \geq b_n$ for all $n \in N$. (by 2 and 3)

$\therefore\ (a_n)$ is a monotonic decreasing sequence and (b_n) is a monotonic increasing sequence.

Further, $a_n \geq b_n \geq b_1$ for all $n \in N$.

and $b_n \leq a_n \leq a_1$ for all $n \in N$.

$\therefore\ (a_n)$ is a monotonic decreasing sequence bounded below by b_1 and (b_n) is a monotonic increasing sequence bounded above by a_1.

$\therefore\ (a_n) \to l$ (say) and $(b_n) \to m$ (say)

Now, $a_{n+1} = \frac{1}{2}(a_n + b_n)$.

Taking limit as $n \to \infty$, we get $l = \frac{1}{2}(l + m)$.

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Problem:

Let (a_n) be a sequence of positive terms such that $a_1 < a_2$ and $a_{n+2} = \frac{1}{2}(a_{n+1} + a_n)$. Then show that (a_{2n-1}) is a monotonic increasing sequence and (a_{2n}) is a decreasing sequence and both converge to the common limit $\frac{1}{3}(a_1 + 2a_2)$. Hence deduce that (a_n) converges to the same limit.

Solution:

We have $a_{n+2} = \frac{1}{2}(a_{n+1} + a_n)$ and $a_1 < a_2$ .... (1)

$\therefore\ a_3 = \frac{1}{2}(a_2 + a_1)$ and $a_1 < a_2$.

$\therefore\ a_1 < a_3 < a_2$

Also $a_4 = \frac{1}{2}(a_3 + a_2)$ and $a_3 + a_2$ (by 1 and 2)

$\therefore\ a_3 < a_4 < a_2$

$\therefore\ a_1 < a_3 < a_4 < a_2$ (by 2 and 3)

Proceeding as above, we get $a_1 < a_3 < a_5 < a_6 < a_4 < a_2$ and so on.

$(a_{2n}) \to l$ (say) and $(a_{2n-1}) \to m$ (say).

Now, $a_{2n+2} = \frac{1}{2}(a_{2n+1} + a_{2n})$ (by 1)

Taking limit as $n \to \infty$, we get $l = \frac{1}{2}(m + l)$.

$l = m$.

Now, let $\epsilon > 0$ be given. Since $(a_{2n}) \to l$, there exists $n \in N$ such that $|a_{2n} - l| < \varepsilon$ for all $n \geq n_1$.

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Similarly there exists $n_2 \in N$ such that $|a_{2n-1} - l| < \varepsilon$ for all $n \geq n_2$. Let $m = \max\{n_1, n_2\}$.

Then $|a_n - l| < \epsilon$ for all $n \geq m$.

$\therefore (a_n) \to l$.

Now, $a_{n+2} = \frac{1}{2}(a_{n+1} + a_n)$

$a_{n+1} = \frac{1}{2}(a_n + a_{n-1})$.

…… …… ………… ………. …….

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…… …… ………… ………. …….

$a_4 = \frac{1}{2}(a_3 + a_2)$

$a_3 = \frac{1}{2}(a_2 + a_1)$

Adding, we get $a_{n+2} + \frac{1}{2}a_{n+1} = \frac{1}{2}(a_2 + 2a_2)$

Taking limit as $n \to \infty$, we get.

$l + \frac{1}{2}l = \frac{1}{2}(a_1 + 2a_2)$ (i.e) $l = \frac{1}{3}(a_1 + 2a_2)$

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Problem 12 :

Prove that the sequencea (an) defined by a1=$\sqrt{k}$ and $a_{n+1}=\sqrt{k+a}_n$ where K>0 converges to root of x^2-x-k=0.

Solution:

First to prove $a_n<a_{n+1}$ for all $n\in N$

We prove this problem by induction on n.

$a_2=\sqrt{k+a}_1=\sqrt{k+k}>\sqrt{k}=a_1$

$a_2>a_1$

$\therefore\ a_1<a_2$

Assume the result is true for n=m

$\therefore\ a_m<a_{m+1}$ for some $m\in N$.

To Prove:

The result is true for n=m+1

ie) $a_{m+1}<a_{m+2}$

Now, $a_{m+2}=\sqrt{k+a}_{m+1}>\sqrt{k+a}_m=a_{m+1}$

$\therefore a_{m+1}<a_{m+2}$

$\therefore$ (a_n) is a monotonic increasing sequence

To Prove:

(a_n) is bounded above

Now, $a_{n+1}>a_n$

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$\sqrt{k+a_n} > a_n$

$(\sqrt{k+a_n})^2 > a_n^2$

$a_n^2 - a_n - k < o$

$\therefore a_n$ lies between the root of $x^2-x-k=0$

$\therefore (a_n)$ is bouned above.

Behaviour the monotonic sequence $a_n) \to l$ (say)

Clearly $0 \leq l \leq \infty$

Now $a_{n+1} = \sqrt{k+a_n}$

Taking limit as $n \to \infty$ we get,

$l = \sqrt{k+l}$

$\therefore l^2 - l - k = 0$

$\therefore l$ is the positive root of $x^2-x-k=0$

$\therefore l = \infty$

SOME THOREMS OF LIMITS

Theorem 3.22 :

Cauchy's first limit theorem If $(a_n) \to l$ Then $(a_1+a_2+........+a_n/n) \to l$.

Proof :

Case (i) :

Let $l=0$

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Let $b_n=a_1+a_2+........+a_n/n$

Let $\varepsilon>0$ be given since $(a_n)\to 0$ there exists $m\in N$ such that $|a_n|<\varepsilon/2$ for all $n\geq m$ ....(1)

Let $n\geq m$

$|b_n| = |a_1+a_2+a_3+........+a_m+a_{m+1}+....+a_n/n|$

$\leq|a_1|+|a_2|+.....+|a_m|+|a_{m+1}|+.....+|a_n|/n$

$=k/n+|a_{m+1}+a_{m+2}+......+a_n|/n$

Where $k=|a_1|+|a_2|+....+|a_m$

$<k/n+\varepsilon|/2$ [n-m/n<1] ....(2)

Now since $(k/n)\to 0$ there exists $n_o\in N$ such that $k/n<\varepsilon/2$ for all $n\geq n_o$ ....(3)

Let $n_1=max(m,n_o)$

Then $|b_n|<\varepsilon$ for all $n\geq n_1$ (by&3)

$\therefore$ $(b_n)\to 0$

Case(ii):

$l\neq 0$

Since $(a_n)\to l$, $(a_n-l)\to 0$

$((a_1-l)+(a_2-l)+........+(a_n-l)/n)\to 0$ (by case (i))

$(a_1+a_2+......+a_n-nl/n)\to 0$

$(a_1+a_2+.......+a_n/n-l)\to 0$

$\therefore(a_1+a_2+.......+a_n/n)\to l$

Hence the proof.

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Thoerem 3.23 : (Cesaro's Thoerom)

If $(a_n) \to a$ and $(b_n) \to b$ then $(a_1 bn + a_2 b_{n-1} + \dots\dots\dots\dots + a_n b_1/n) \to ab$.

Proof :

Let $c_n = (a_1 bn + a_2 b_{n-1} + \dots\dots\dots\dots + a_n b_1/n)$

Put $a_n = a + r_n$ so that $(r_n) \to 0$

Then $c_n = ((a+r_1)b_n + (a+r_2)\, b_{n-1} + \dots\dots\dots\dots + (a+r_n)\, b_1/n)$

$= a(b_1 + b_2 + \dots\dots\dots\dots + b_n)/n + r_1 b_n + \dots\dots\dots\dots + r_n b_1/n$

Now By Cauchy first limit thorem

$(b_1 + b_2 + \dots\dots\dots\dots + b_n)/n \to b$

$a(b_1 + b_2 + \dots\dots\dots\dots + b_n)/n \to ab$

Hence it is enough if we prove that

$r_1 b_n + \dots\dots\dots\dots + r_n b_1/n \to 0$

Now since $(b_n) \to b$, (b_n) is a bounded sequence.

$\therefore$There exists a real number $k>0$ such that $|b_n| \leq k$ for all n.

$| r_1 b_n + \dots\dots\dots\dots + r_n b_1/n| \leq |r_1 + r_2 + \dots\dots + r_n|/n$

Since $(r_n) \to 0$, $(r_{1n} + \dots\dots\dots\dots + r_{n1}/n) \to 0$.
$(r_1 b_n + \dots\dots\dots\dots + r_n b_1/n) \to 0$

Hence the proof.

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Cauchy's & Second limit theorem

Theorem 3.24:

Let (a_n) be a sequence of positive terms. Then lim $a_n^{1/n}$=$\lim_{n\to\sim} a_{n+1}/a_n$ Provided the limit on the right hand side exist, whether finite (or) infinite.

Proof:

Case(i):

$\lim_{n\to\infty} a_{n+1}/a_n = l_1$ finite

Let €> 0 be any given real number.

Then there exist m € N. Such that l - 1/2 € < an+1/an < l+2 € for all n ≥ m.

Now, choose n ≥m,

l - 1/2 €< am+1/am < l+1/2€

l - 1/2 €< am+2/am < l+1/2€

:::::--------:::::-------::::::::::

l -1/2 €< an/an-1 < l+1/2€

Multiplying these inequalities we obtain,

(l -1/2€) $^{n-m}$ < a_n/a_m < (l+1/2€) $^{n-m}$

am (l-1/2€) ^{n}/(l-1/2)^{m} < an < am (l+1/2€)^{n}/ (l+/2€)^{m}

k1 (l-1/2€) ^{n} < an < k2 (l+1/2€)^{n}

Where k1,k2 are same constants.

Therefore, k1$^{1/n}$ (l-1/2€) < an $^{1/n}$ < k2 $^{1/n}$ (l+1/2€)

Therefore, now, (k1$^{1/n}$ (l-1/2€)) -> l- 1/2€.

[(k1)$^{1/n}$ -> 1]

Therefore, there exist n1 € N such that

(l-1/2€) - 1/2€ < k1$^{1/n}$ (l-1/2€) < (l-1/2€) + 1/2€ ∀ n ≥ n1

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$k1 (l-1/2€)^n < an < k2 (l+1/2€)^n$

Where k1,k2 are same constants.

Therefore, $k1^{1/n} (l-1/2€) < an^{1/n} < k2^{1/n} (l+1/2€)$

Therefore, now, $(k1^{1/n} (l-1/2€)) \rightarrow l- 1/2€$. $[(k1)^{1/n} \rightarrow l]$

Therefore, there exist n1 € N such that $(l-1/2€) - 1/2€ < k1^{1/n} (l-1/2€) < (l-1/2€) + 1/2€ + n \geq n1$.

$l-€<k1^{1/n}(l-€/2)<l$

There exists n_2€N such that,

$(l+1/2€) - 1/2€ < k2^{1/n} (l+1/2€) < (l+1/2€) + 1/2€ + n \geq n1$.

Let m=max{n1,n2}

Then $l- €<k1^{1/n} (l-1/2€) < an^{1/n} < k2^{1/n} (l+1/2€)<l+0, n\geq n0$

$l- €< an^{1/n} < l+€^{1/n}$ hence$(a_n^{1/n})$

Case(ii):

$\lim_{n\rightarrow\infty} a_{n+1}/a_n=\infty$

Then $\lim_{n\rightarrow\infty} (1/a_{n+1})/(1/a_n)=\infty$

by case (i) $(1/a_n)\rightarrow \infty$

$(a_n)^{1/n}\rightarrow\infty$

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Theorem 3.25:

Let (a_n) be any sequence and $\lim_{n\to\infty} |a_n / a_{n+1}| = l$ if $l > 1$ then $(a_n) \to 0$

Proof:

Let K be any real no, such that $|<K<l|$

Since $\lim_{n\to\infty} |a_n / a_{n+1}| = l$, there exists $m \in N$ such that $l - \epsilon < |a_n / a_{n+1}| < l + \epsilon$

Choosing $\epsilon = l - k$ we obtain $|a_n / a_{n+1}| > k$

Now fix $n \geq m$. then ,

$|a_m / a_{m+1}| > k$; $|a_{m+1} / a_{m+2}| > k$; $|a_{n-1} / a_{n>k}|$

Multiplying the above inequalities we get,

$|a_m / a_n| > k^{n-m}$

$|a_n / a_m| < k^m (1/k)^n$

$|a_n| < k^m |a_m| (1/k)^n$

$|a_n| < Ar^n$ where $A = |a_m| k^m$ is a constant and $r = 1/k$

Now $k > 1 \to 0 < r < 1$.

$(a^n) \to 0$

The above theorem is true even if $l = \infty$.

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Theorem 3.26:

Let (a_n) be any sequence of the terms and $\lim_{n\to\infty} (a_n / a_{n+1})=l$. If $l<1$ then $(a_n)\to 0$

Proof:

Let K be any real no, such that $|<K<l|$

Since $\lim_{n\to\infty} |a_n / a_{n+1}|=l$, there exists m€N such that $l-€< |a_n / a_{n+1}|<l+€$

Choosing $€=l-k$ we obtain $|a_n / a_{n+1}|>k$

Now fix $n\geq m$. then ,

$|a_m / a_{m+1}|>k$; $|a_{m+1} / a_{m+2}|>k$; $|a_{n-1} / a_{n>k}|$

Multiplying the above inequalities we get,

$|a_m / a_n|>k^{n-m}$

$|a_n / a_m|<k^m (1/k)^n$

$|a_n |<k^m |a_m|(1/k)^n$

$|a_n |<Ar^n$ where $A=|a_m|k^m$ is a constant and $r=1/k$

Now $k> 1\to 0< r < 1$.

$(a^n)\to 0$

The above theorem is true even if $l=\infty$.

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Theorem 3.27:

If the sequences (a_n) and (b_n) converge to 0 and (b_n) is strictly monotonic decrasing then $\lim_{n\to\infty} (a_n / b_n) = \lim_{n\to\infty} (a_n - a_{n+1} / b_n - b_{n+1})$ provided the limit on the right hand side exists whether finite or infinite.

Proof:

Case(i):

Let $\lim_{n\to\infty} (a_n - a_{n+1} / b_n - b_{n+1}) = l$, finite let $\epsilon > 0$ be given . then there exists $m \in N$ such that ,

$L1 - \epsilon < a_n - a_{n+1} / b_n - b_{n+1} < l + \epsilon$

Since $b_n - b_{n+1} > 0$, we get

$(b_n - b_{n+1})(l - \epsilon) < a_n - a_{n+1} < (b_n - b_{n+1})(l + \epsilon)$

Let $n > p \geq m$

Then $(b_p - b_{p+1})(l - \epsilon) < a_p - a_{p+1} < (b_n - b_{n+1})(l + \epsilon)$

$(b_{p+1} - b_{p+2})(l - \epsilon) < b_{p+1} - b_{p+2} < (a_{p+1} - a_{p+2})(l + \epsilon)$

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$(b_{n-1}-b_n)(l-€) < a_{n-1}-a_n < (b_{n-1}-b_n)(l+€)$

Adding the above inequalites we get,

$(b_p-b_n)(l-€) < a_p-a_n < (b_p-b_n)(l-€)$

Taking limit as n->∞; we get ,

$(b_p)(l-€) < a_p < b_p(l+€)$ (therefore (a_n) , (b_n)->0)

Therefore $l-€< a_p/b_p < l+€$

Therefore $|a_p/b_p-i| < €$

Therefore $\lim_{n\to\infty}(a_n/b_n)=l$

Case(ii):

$\lim_{n\to\infty}(a_n- a_{n+1}/ b_n- b_{n+1})=\infty$

Let k > 0 be any real no. then there exists m € N such that $a_n- a_{n+1}/ b_n- b_{n+1} > k$ for all n ≥m.

Therefore $a_n- a_{n+1}> (b_n- b_{n+1})k$

let n >p ≥m

writing the inequalities for n=p,p+1,……..,n and adding we get ,

$a_p- a_n > k(b_p- b_n)$

Taking limit as n->∞, we get

Therefore $a_p/b_p \geq k$ for all p≥m

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$\lim_{n\to\infty} (a_n - a_{n+1} / b_n - b_{n+1}) = \infty$

Let $k > 0$ be any real no. then there exists $m \in N$ such that $a_n - a_{n+1} / b_n - b_{n+1} > k$ for all $n \geq m$.

Therefore $a_n - a_{n+1} > (b_n - b_{n+1}) k$

let $n > p \geq m$

writing the inequalities for $n=p,p+1,\ldots\ldots..,n$ and adding we get ,

$a_p - a_n > k (b_p - b_n)$

Taking limit as $n\to\infty$, we get

Therefore $a_p/b_p \geq k$ for all $p \geq m$

Therefore (a_n /b_n) converges to ∞

Note :

The above theorem is true ever if $l=\infty$

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Problem 1:

Show that $\lim_{n\to\infty} \frac{1}{n}(1+\frac{1}{2}+......+\frac{1}{n})=0$

Solution:

Let $a_n=\frac{1}{n}$

We know that $(a_n)\to 0$

Hence by Cauchy's first limit theorem, we get

$(\frac{a1+a2+\cdots+an}{n})\to 0$

$\therefore \frac{1}{n}(1+\frac{1}{2}+......+\frac{1}{n})\to 0$

$\lim_{n\to\infty} \frac{1}{n}(1+\frac{1}{2}+ .... +\frac{1}{n})=0$ is proved

Problem 2:

Show *that* $\lim_{n\to\infty} n^{\frac{1}{n}}= 1$

Solution:

Let $a_n=n$

$a_{n+1}=n+1$

$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (\frac{n+1}{n})$

$= \lim_{n\to\infty} (1 + \frac{1}{n})=1$

$\therefore$ By Cauchy 's second limit theorem we get

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$$\lim_{n\to\infty} n^{1/n}=1$$

Hence the proof

Problem 3:

Prove that $\frac{1}{n}[(n+1)(n+2)\ldots.(n+n)]^{\frac{1}{n}} \to \frac{4}{e}$

Solution:

Let $a_n=\frac{1}{n}[(n+1)(n+2)\ldots\ldots(n+n)]^{\frac{1}{n}}$

$=[\frac{(n+1)(n+2)\ldots\ldots(n+n)}{n^n}]^{1/n}$

$=[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots(1+\frac{n}{n})]^{\frac{1}{n}}$

Let $b_n=(1+\frac{1}{n})(1+\frac{2}{n})\ldots\ldots.(1+\frac{n}{n})$

So that $a_n=b_n^{\frac{1}{n}}$

$b_{n+1}=(1+\frac{1}{n+1})(1+\frac{2}{n+1})\ldots\ldots(1+\frac{n+1}{n+1})$

now $\frac{b_{n+1}}{b_n}=\frac{\left(1+\frac{1}{n+1}\right)\left(1+\frac{2}{n+1}\right)\ldots\ldots(1+\frac{n+1}{n+1}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots(1+\frac{n}{n})}$

$=(2n+1)(2n+2)\frac{n^n}{(n+1)^{n+2}}$

$=\frac{2(2n+1)}{n+1}.\frac{n^n}{(n+1)^n}$

$=2(\frac{2+\frac{1}{n}}{1+\frac{1}{n}}).\frac{1}{(1+\frac{1}{n})^n}$

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$$\therefore \left(\frac{b_{n+1}}{b_n}\right) \to \frac{4}{e}$$

By Cauchy 's second limit theorem we get

$$(b_n^{\frac{1}{n}}) \to \frac{4}{e}$$

$$\therefore (a_n) \to \frac{4}{e}$$

$\therefore \frac{1}{n}[(n+1)(n+2)\ldots\ldots(n+n)^{\frac{1}{n}}] \to \frac{4}{e}$ is proved

Problem 4:

Prove that $\lim\limits_{n\to\infty} \frac{x^n}{n!}=0$

solution:

Let $a_n = \frac{x^n}{n!}$

$$a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{x^n}{n!}\frac{(n+1)!}{x^{n+1}} = \frac{n+1}{x}$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \infty$$

We know that $(a_n) \to 0$

$\therefore \lim\limits_{n\to\infty} \frac{x^n}{n!} = 0$ is proved

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Problem 5:

Show that $\lim_{n\to\infty} n!/n^n=0$

Solution:

Let $a_n= n!/n^n$

$| a_n/ a_{n+1}|= n!/n^n \, (n+1)^{n+1}/(n+1)!$

$=(n+1/n)^n$

$=(1+1/n)^n$

$\lim_{n\to\infty}| a_n/ a_{n+1}| = \lim_{n\to\infty}(1+1/n)^n$

$=e$ (by problem 3 of 3.7)

>1

Therefore $(a_n)\to 0$(by theorem 3.25)

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Subsequences

Definition:

Let(a_n) be a sequences. Let (n_k) be a strictly increasing sequence of natural numbers. Then (a_{n_k}) is called a subsequences of (a_n).

Note:

The terms of a subsequence occur in the same order in which they occur in the original sequence.

Examples:

1. (a_{2n}) is a subsequences of any sequences(a_n). Note that in this example the internal between any two terms of the subsequence is the same, (i.e) n1=2, n2=4, n3=6, nk=2k
2. (a_{k^2}) is a subsequence of any sequence (a_n) hence a_{n_1}=a1, a_{n_2}=a4, a_{n_3}=a9.......... here the interval between two successive terms of the subsequence goes on increasing as k becomes large. Thus the interval between various terms of a subsequence need not be regular.
3. Any sequence(a_n) is a subsequence of itself.
4. Consider the sequence(a_n) given by 1,0,1,0.... Now, (b_n), given by 1,1,1..... is a subsequence of (a_n). hence (a_n) is not converges to 1. Thus a subsequence of non-convergent sequence can be a convergent sequence.

Note :

A subsequence of a given subsequence (a_{n_k}) of a sequence (a_n) is again a subsequence of (a_n).

Theorem 3.28:

If a sequence (a_n) converges to l. then every subsequence (a_{n_k}) of (a_n) also converges to l.

Proof:

Let $\epsilon > 0$ be given.

Since (a_n)->l there exists $m \in N$ such that

$|a_n - l| < \epsilon$ for all $n \geq m$

Now choose $n_k \geq m$.

Then $k \geq k_0 \rightarrow n_k \geq n_{k_0}$

$n_k \geq m$

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$|a_n - l| < \epsilon$ (by (1))

Thus $|a_n - l| < \epsilon$ for all $k \geq k_0$

Therefore $(a_{nk}) \rightarrow l$

Note 1:

If a subsequence of a sequence converges then the original sequence need not converges (refer examples 4)

Note 2:

If a sequence(a_n) has two subsequence converges to two different limit, then (a_n) close not converge for example. Consider the sequence(a_n) given by

A_n={yn if n is even , 1+1/n if n is odd

Here the subsequence$(a_{2n}) \rightarrow 0$ and the subsequence $(a_{2n-1}) \rightarrow 1$. Hence the given sequence (a_n) does not converge.

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THEOREM: 3.29

If the subsequence (a_{2n-1}) and (a_{2n}) of a sequence (a_n) converge to the same limit l then (a_n) also converges to

SOLUTION:

Let $\varepsilon > 0$ be given. Since $(a_{2n-1}) \to l$ there exists $n_1 \in N$ such that $|a_{2n-1}-l| < \varepsilon$ for all $2n-1 \geq n_1$.

Similarly there exists $n_2 \in N$ such that $|a_{2n}-l| < \varepsilon$ for all $2n \geq n_1$.

Let $m = \max\{n_1, n_2\}$.

Clearly $|a_n-l| \leq$ for all $n \geq m$.

$\therefore (a_n) \to l$.

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NOTE :

The above result is true even if we have $l=\infty$ or $-\infty$

DEFINITION:

Let (a_n) be k sequence. A natural number m is called a peak point of the sequence (a_n) if $a_n < a_m$ for all $n>m$.

EXAMPLE:

1. For the sequence $(1/n)$, every natural number is a peak point and hence the sequence has infinite number of peak points. In general for k strictly monotonic decreasing sequence every natural number is a peak point.
2. Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, -1, -1, \ldots\ldots$ Here 1,2,3

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are the peak point of the sequence.

3. The sequence 1, 2, 3, ... has no peak point. In general k monotonic increasing sequence has no peak point.

THEOREM: 3.30

Every sequence (a_n) has a monotonic subsequence.

PROOF:

Case(i)

(a_n) has infinite number of peak points.

Let the pea point be $n_1 < n_2 < \ldots\ldots \leq n_k < \ldots\ldots$

Then $a_{n1} > a_{n2} > \ldots\ldots > a_{nk} > \ldots\ldots$

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∴ (a_{nk}) is a monotonic decreasing subsequence of (a_n).

Case(ii):

(a_n) has only k finite number of peak points or no peak point.

Choose a natural number n_1 such that there is no peak point greater than or equal to n_1. Since is n_1 is not a peak point of (a_n), there exists $n_2 > n_1$ such that $a_{n2} \geq a_{n1}$. Again since n_2 is not a peak point, there exists $n_3 > n_2$ such that $a_{n3} \geq a_{n2}$.

Repeation this process we get a monotonic increasing subsequence (a_{nk}) of (a_n).

THEOREM: .3.31

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Every bounded sequence has a convergent subsequence.

PROOF:

Let (a_n)be a bounded sequence . let (a_{nk}) be k monotonic subsequence of (a_n).

Since (a_n) is bounded (a_{nk}) is also bounded.

∴ (a_{nk}) is k bounded monotonic sequence and hence converges .

∴ (a_{nk}) is k convergent subsequence of (a_n).

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3.10 LIMIT POINTS

Definition. Let (a_n) be a sequence of real numbers a is called a limit point or a cluster point of the sequence (a_n) if given $\varepsilon > 0$, there exists infinite number of terms of the sequence in $(a - \varepsilon, a + \varepsilon)$. If the sequence (a_n) is not bounded above then ∞ is a limit point of the sequence. If (a_n) is not bounded below then $-\infty$ is a limit point of the sequence.

Examples.

1. *Consider the sequence 1 , 0 , 1 , 0 , ... For this sequence 1 is a limit*

point since given $\varepsilon > 0$, the interval $(1 - \varepsilon, 1 + \varepsilon)$ contains infinitely many terms $a_1, a_3, a_5, \ldots\ldots$ of this sequence. Similarly 0 is also a limit point of this sequence.

2. *If a sequence (a_n) converges to l then l is a point of the*

Sequences. For, given $\varepsilon > 0$, there exists m∈N such that $a_n \in (1 - \varepsilon, 1 + \varepsilon)$ for all

$n \geq m$.

$\therefore$ *$(l - \varepsilon, l + \varepsilon)$ contains infinitely many terms of the sequences.*

3. *The sequences (a_n)=1, 2, 3, ... n . . . . is not bounded*

above and hence ∞ is a limit point.

4. *The sequence (a_n)=1, -1, 2, -2, ... n, -n , ... is*

Neither bounded above nor bounded below. Hence ∞ and $-\infty$ are limit points of the sequence.

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Theorem 3.32

Let (a_n) be a sequence. A real number a is a limit point of (a_n) iff there exists a subsequence (a_{nk}) of (a_n) converging to a.

Proof. *Suppose there exists a subsequence (a_{nk}) of (a_n) converging to a.*

Let $\varepsilon > 0$ be given . Then there exists $k_0 \in N$ such that

$a_{nk} \in (a - \varepsilon, a + \varepsilon)$ *for all* $k \geq k_0$.

$\therefore (a - \varepsilon, a + \varepsilon)$ *contains infinitely many terms of the sequence* (a_n).

$\therefore$ *a is a limit point of the sequence* (a_n).

Conversely suppose a is a limit point of (a_n).

Then for each $\varepsilon > 0$ the interval $(a - \varepsilon, a + \varepsilon)$ contains infinitely many terms of the sequence. In particular we can find $n_1 \in N$ such that

$a_{n1} \epsilon (a - 1; a + 1)$.

Also we can find $n_2 > n_1$ such that $a_{n2} \in \left(a - \frac{1}{2}, a + \frac{1}{2}\right)$

Proceeding like this we can find natural numbers $n_1 < n_2 < n_3 \ldots.$ *such that* $a_{nk} \in \left(a - \frac{1}{k}, a + \frac{1}{k}\right)$.

Clearly (a_{nk}) is a subsequence of (a_n) and $|a_{nk-a}| < \frac{1}{k}$.

For any $\varepsilon > 0, |a_{nk-a}| < \varepsilon$ *if* $k > \frac{1}{\varepsilon}$.

$\therefore (a_{nk}) \to a.$

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Theorem 3.33

Every bounded sequence has at least one limit point.

Proof. *Let (a_n) be a bounded sequence. Then there exists a convergent subsequence (a_{nk}) of (a_n) converging to l. Hence l is a limit point of (a_n).*

Note. *In general every sequence (a_n) has at least one limit point (finite or infinite).*

Theorem 3.34

A sequence (a_n) converges to l iff (a_n) is bounded and l is the only limit point of the sequence.EEE

Proof. *Let $(a_n) \to l$. Then (a_n) is bounded.*

Also l is a limit point of the sequence (a_n).

Now suppose l_1 is any other limit point of (a_n). Then there exists a subsequence (a_{nk}) of (a_n) such that $(a_{nk}) \to l_1$.

Now, since $(a_n) \to l$, we have $(a_{nk}) \to l$.

$\therefore l = l_1$.

Thus l is the only limit point of the sequence.

Conversely, suppose l is the only limit point of (a_n). Suppose (a_n) dose not converge to l. Then there exists at least one $\varepsilon > 0$ such that infinitely many terms of the sequence lie outside $(l - \varepsilon, l + \varepsilon)$.Hence we can find a subsequence (a_{nk}) of (a_n) such that a_{nk} Ɇ $(l - \varepsilon, l + \varepsilon)$ for all k.

Since (a_n) is a bounded sequence, (a_{nk}) is also a bounded sequence. Hence (a_{nk}) has also a limit point by theorem 3.33, say l' and $l' \neq l$.

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$\therefore (a_n)$ has two limit points l and l which is a contradiction.
Hence $(a_n) \to l$.

CAUCHY SEQUENCE

Definition. A sequence (a_n) is said to be a Cauchy sequence if given $\varepsilon > 0$, there exists $n_0 \in N$ such that $|a_n - a_m| < \varepsilon$ for all n, m $m \geq n_0$.

Note. In the above definition the condition $|a_n - a_m| < \varepsilon$ for all n, $m \geq n_0$ can be written in the following equivalent form, namely, $|a_{n+p} - a_n| < \varepsilon$ for all $n \geq n_0$ and for all positive integers p.

Examples.

1. The sequence $\left(\frac{1}{n}\right)$ is a Cauchy sequence.

Proof. **Let $(a_n) = \left(\frac{1}{n}\right)$. Let $\varepsilon > 0$ be given.**

Now, $|a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right|$.

$\therefore$ If we choose n_0 to be any positive integer greater than we get $|a_n - a_m| < \varepsilon$ for all n, $m \geq n_0$.

$\therefore \left(\frac{1}{n}\right)$ is a Cauchy sequence.

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Example: 2

The sequence $[(-1)^n]$ is not a cauchy sequence

Proof:

Let $(a_n) = \{ (-1)^n \}$

$\therefore |a_n - a_{n+1}| = 2$

$\therefore$ If $\varepsilon < 2$, we cannot find n_0, such that $|a_n - a_{n+1}| < \varepsilon$

for all $n \geq n_0$

$\therefore [(-1)^n]$ is not a cauchy sequence .

Example: 3

(n) is not a cauchy sequence

Proof:

Let $(a_n) = (n)$

$\therefore |a_n - a_m| \geq 1$ if $n = m$

$\therefore$ If we choose $\varepsilon < 1$,we cannot find n_0 such that

$|a_n - a_m| < \varepsilon$ for all $n,m \geq n_0$. (n) is not a cauchy

sequence.

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Theorem 3:35

Any convergent sequence is a cauchy sequence .

Proof:

Let $(a_n) \to l$. Then given $\varepsilon < 0$, there exists $n_0 \varepsilon N$ such that $|a_n - l| < \frac{1}{2}\varepsilon$ for all $n \geq n_0$.

$\therefore |a_n - a_m| = |a_n - l + l - a_m|$

$\leq |a_n - l| + |l - a_m|$

$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$ for all $n,m \geq n_0$.

$\therefore (a_n)$ is a cauchy sequence.

Theorem: 3:36

Any cauchy sequence is abounded sequence.

Proof:

Let(a_n) be a cauchy sequence.

Let $\varepsilon > 0$ be given ,then there exists $n_0 \varepsilon N$ such that

$|a_n - a_m| < \varepsilon$ for all $n,m \geq n_0$.

$\therefore |a_n| < |a_{n0}| + \varepsilon$ for all $n \geq n_0$.

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Now, let $k = \max\{|a_1|, |a_2|, \ldots\ldots |a_{n0}| + \varepsilon\}$

Then $|a_n| \leq k$ for all n . Hence (a_n) is a bounded

sequence

Theorem:3:37

Let (a_n) be a cauchy sequence .If (a_n) has a subsequence (a_{nk}) convering to l,then $(a_n) \rightarrow l$

Proof:

Let $\varepsilon < 0$ be given ,then there exists $n_0 \varepsilon N$ such that

$|a_n - a_m| < ½ \varepsilon$ for all $n,m \geq n_0$ → ①

Also since $(a_{nk}) \rightarrow l$,there exists $k_0 \varepsilon N$ such that

$|a_{nk} - l| < ½ \varepsilon$ for all $k \leq k_0$ → ②

Choose n_k such that $n_k \geq n_{k0}$ and n_0

Then $|a_n - l| = |a_n - a_{nk} + a_{nk} - l|$

$\leq |a_n - a_{nk}| + |a_{nk} - l|$

$< ½ \varepsilon + ½ \varepsilon = \varepsilon$ for all $n \geq n_0$.

Hence $(a_n) \rightarrow l$.

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Theorem:3:38

(cauchy's general principal of convergence). A sequence (a_n) in R is convergent iff it is a cauchy sequence.

Proof:

Let $(a_n) \to l$, then given $\varepsilon > 0$, there exists $n_0 \varepsilon N$ such that $|a_n - l| < \frac{1}{2}\varepsilon$ for all $n \geq n_0$.

$$\therefore |a_n - a_m| = |a_n - l + l - a_m|$$
$$\leq |a_n - l| + |l - a_m|$$
$$< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon \text{ for all } n,m \geq n_0.$$

$\therefore (a_n)$ is a cauchy sequence ,that any convergent Sequence is a cauchy sequence.

Conversely,let (a_n) be a cauchy sequence in R. (a_n) is a bounded sequence,we know that "Any Cauchy sequence is a bounded sequence".

$\therefore$There exists a subsequence (a_{nk}) of (a_n) such that $(a_{nk}) \to l$,we know that "Every bounded sequence has a convergent sequence".

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