ABSTRACT ALGEBRA - II

UNIT - IV

ELEMENTARY TRANSFORMATIONS

Definition:

Let A be an m×n matrix over a field F. An elementary row-operation on A is of any one of the following three types.

- 1. The interchange of any two rows.
- 2. Multiplication of a row by a non-zero element c in F.
- 3. Addition of any multiple of one row with any other row.

Similarly we define an elementary column operation on A as any one of the following three types.

- 1 .The interchanges of any two columns.
- 2. Multiplication of a column by a non-zero element c in F.
- 3. Addition of any multiple of one column with any other column.

Example:

Let $A=\begin{pmatrix}1 & 2\\ 2 & 1\\ 3 & -1\end{pmatrix}$, $A_1=\begin{pmatrix}3 & -1\\ 2 & 1\\ 1 & 2\end{pmatrix}$, $A_2=\begin{pmatrix}2 & 2\\ 4 & 1\\ 6 & -1\end{pmatrix}$, $A_3=\begin{pmatrix}1 & 2\\ 5 & 7\\ 3 & -1\end{pmatrix}$.

A_1 is obtained from A by interchanging the first and third rows.

A_2 is obtained from A by multiplying the first column of A by 2.

A_3 is obtained from A by adding to the second row the multiple by 3 of the first row.

Notation:

We shall employ the following notations for elementary transformation.

I. Interchange of i^{th} and j^{th} rows will be denoted by $R_i \leftrightarrow R_j$

II. Multiplication of i^{th} row by a non-zero element $c \in F$ will be denoted by $R_i \to cR_i$.

III. Addition of k times the j^{th} row to the i^{th} row will be denoted by $R_i \to R_i + kR_j$.

The corresponding column operations will be denoted by writing C in the place of R.

Definition:

An m×n matrix B is said to be row equivalent (column equivalent) to an m×n matrix A if B can be obtained from A by a finite succession of elementary row operations (column operations).

A and B are said to be equivalent if B can be obtained from A by a finite succession of elementary row or column operations.

If A and B are equivalent. We write $A \sim B$.

Definition:

A matrix obtained from the identity matrix by applying a single elementary row or column operation is called an elementary matrix.

For example, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ are elementary matrices obtained from the identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ by applying the elementary operations $R_1 \leftrightarrow R_2$, $R1 \to 4R_1$, $R_3 \to R_3 + 2R_2$ respectively.

Theorem:7.23

Any elementary matrix is non -singular.

Proof:

The determinant of the identity matrix of any order is1. Hence the determinant of an elementary matrix obtained by interchanging any two rows is-1. The determinant of an elementary matrix obtained by multiplying any row by $k \neq 0$ is k. The determinant of an elementary matrix obtained by adding a multiple of one row with another row is1.Hence any elementary matrix is non -singular.

Theorem:7.24

Let A be an mxn matrix and B be an nxp matrix.Then every elementary row(column) operation of the product AB can be obtained by subjecting the matrix A(matrix B)to the same elementary row(column)operation.

Proof:

Let R1,R2,……Rm denote the rows of the matrix A and C1,C2,….,Cp denote the column of B. By the definition of matrix multiplication.

$$AB=\begin{bmatrix} R1C1 & R1C2 & \cdots & R1Cp \\ \vdots & & \vdots & \vdots \\ RmC1 & RmC2 & \cdots & RmCp \end{bmatrix}$$

It is obvious from the above representation of AB that if we apply any elementary row operation on A the matrix AB is also subjected to the same elementary row operation. Also if we apply any elementary column operation on B the matrix AB is also subjected to the same elementary column operation.

Theorem:7.25

Each elementary row operation on an mxn matrix A is equivalent to pre -multiplying the matrix A by the corresponding elementary mxm matrix.

Proof:

Since A is an mxn matrix we can write A=IA where I is the identity matrix of order m. By thm 7.24 an elementary row operation IA is equivalent to the same row operation on I. But an elementary row operation on I gives an elementary matrix. Hence by pre- multiplying A by the corresponding elementary matrix we get the required row operation on A.

Note.

Similarly each elementary column operation of an mxn matrix A is equivalent to post multiplying the matrix A by the corresponding elementary nxn matrix.

Corollary 1.

If two mxn matrices A and B are row equivalent then A=PB where P is a non -singular mxm matrix.

Proof:

Since A is row equivalent to B, A can be obtained from B by applying successive elementary now operations. Hence A=E1E2…….EnB where each Ei is an elementary matrix. Since each Ei is non -singular. A =PB where P=E1E2…..En and p is non- singular.

Corollary 2.

If two matrices A and B are column equivalent then A =BQ where Q isa non -singular matrix.

Corollary 3.

If two mxn matrices A and B equivalent then A =BQ where p is a non-singular. mxm matrix and Q is a non- singular nxn matrix.

Corollary 4.

The inverse of an elementary matrix is again an elementary matrix.

Proof:

Let E be an elementary matrix obtained from I by applying some elementary operations. If we apply the reverse operation on E, then E is carried back to I. Let E*be the elementary matrix corresponding to the reverse operation.

Then $E^*E = EE^* = I$. Hence $E^* = E^{-1}$

Hence E^{-1} is also an elementary matrix.

Canonical form of a matrix:

We now use elementary row and column operations to reduce any matrix to a simple form, called the *canonical form of a matrix*.

Theorem 7.26:

By successive applications of elementary row and column operations, any non-zero m $\times$ n matrix A can be reduced to a diagonal matrix D in which the diagonal entries are either 0 or 1 and all the 1's proceeding all the zeros on the diagonal. In otherwords, any non-zero m $\times$ n matrix is equivalent to a matrix of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where I_{r} is the r $\times$ r identity matrix and 0 is the zero matrix.

Proof:

We shall prove the theorem by induction on the number of rows of A. Suppose a has just one row.

Let A = $(a_{11}a_{12}\ldots\ldots.a_{1n})$.

Since A $\neq$ 0, by interchanging columns, if necessary, we can bring a non-zero entry c to the position a_{11}.

Multiplying A by c^{-1} we get 1 as the first entry.

Other entries in A can be made zero by adding suitable multiples of 1. Thus the result is true when m = 1

Now, suppose that the result is true for any non-zero matrix with m $-$ 1 rows.

Let A be a non-zero m $\times$ n matrix. By permuting rows and columns we can bring some non-zero entry c to the position a_{11}.

Multiplying the first row by c^{-1} we get 1 as the first entry.

All other entries in the first column can be made zero by adding suitable multiples of the first row to each other row.

Similarly all the other entries in the first row can be made zero.

This reduces A to a matrix of the form B = $\begin{bmatrix} I_1 & 0 \\ 0 & C \end{bmatrix}$ where C is the $(m - 1) \times (m - 1)$ matrix.

Now by induction hypothesis C can be reduced to the desired form by elementary row and column operations.

Hence A is equivalent to a matrix of the required form.

Corollary: 1

If A is an $m \times n$ matrix there exist non-singular square matrices P and Q of order m and n respectively such that PAQ = $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

The result follows from corollary 3 of the theorem 7.25

Corollary: 2

Any non-singular square matrix A of order n is equivalent to the identity matrix.

Proof:

By corollary 1, PAQ = $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Since P, A, Q are all non-singular $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is non-singular.

This is possible iff $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = I_n$.

Corollary: 3

Any non-singular matrix A can be expressed as a product of elementary matrices.

Proof:

By corollary 2, PAQ = I_n

Hence A = $P^{-1}Q^{-1}$.

Further by corollary 4 of theorem 7.25, P^{-1} and Q^{-1} are products of elementary matrices.

Hence A is a product of elementary matrices.

Note:

The inverse of anon-singular matrix A can be computed by using elementary transformations. Let A be a non-singular matrix of order n. Then $AA^{-1} = A^{-1}A = I$. Now the non-singular matrix A^{-1} can be expressed as a product of elementary matrices.

Let $A^{-1} = E_1E_2 \ldots\ldots E_n$.

Then $I = A^{-1}A = E_1E_2 \ldots\ldots E_nA$.

Thus every non-singular matrix A can be reduced to I by pre-multiplying A by elementary matrices.

Hence A can be reduced to the identity matrix by applying successive elementary row operations.

Now, $A = I\,A$. Reduce the matrix A in the left hand side to I by applying successive elementary row operations and apply the same elementary row operations to the factor I in right hand side.

Then we get $I = B\,A$ so that $B = A^{-1}$

Solved problem

Problem 1. Reduce the matrix A = $\begin{bmatrix}1 & 2 & -1\\ 1 & 1 & 2\\ 2 & 4 & -2\end{bmatrix}$ to the canonical form.

Solution.

$$A = \begin{bmatrix}1 & 2 & -1\\ 1 & 1 & 2\\ 2 & 4 & -2\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & -1\\ 0 & -1 & 3\\ 0 & 0 & 0\end{bmatrix} \quad R_2\to R_2- R_1 \,;\, R_3\to R_3 -2R_1$$

$$\sim \quad \sim\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix} \quad C_3\to C_3 +3C_2 \,;\, C_3\to C_3 +C_1$$

$$\sim\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} \quad R_2\to - R_2$$

Problem 2. Find the inverse of the matrix A= $\begin{bmatrix}1 & 0 & 2\\ 3 & 1 & -1\\ -2 & 1 & 3\end{bmatrix}$

Solution.

$$\begin{bmatrix}1 & 0 & 2\\ 3 & 1 & -1\\ -2 & 1 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad A$$

$$\to \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & -7\\ 0 & 1 & 7\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ 2 & 0 & 1\end{bmatrix} A,\ R_2\to R_2 -3R_1 \,;R_3\to R_3 +2R_1$$

$$\rightarrow\begin{bmatrix}1 & 0 & 2\\0 & 1 & -7\\0 & 0 & 14\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\-3 & 1 & 0\\5 & -1 & 1\end{bmatrix}A,\quad R_3\rightarrow R_3-R_2$$

$$\rightarrow\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{2}{7} & \frac{1}{7} & \frac{-1}{7}\\\frac{-1}{2} & \frac{1}{2} & \frac{1}{2}\\\frac{5}{14} & \frac{-1}{14} & \frac{1}{14}\end{bmatrix}\quad R_1\rightarrow R_1-\frac{1}{7}R_3\ ;\ R_2+\frac{1}{2}R_3\ ;\ R_3\rightarrow\frac{1}{14}R_3$$

$$\rightarrow A^{-1}=\begin{bmatrix}\frac{2}{7} & \frac{1}{7} & -\frac{1}{7}\\-\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\\frac{5}{14} & -\frac{1}{14} & \frac{1}{14}\end{bmatrix}$$

Definition: Let A and b be two square matrices of order n. B said to be **similar** to A if there exists a $n\times n$ non – singular matrix P such that $B=P^{-1}AP$.

Solved Problems

Problem 1 Similarity of matrices is an equivalence relation in the set of all $n\times n$ matrices.

Proof

Let S be the set of all $n\times n$ matrices.

Let $A \in S$.

Since $A=I^{-1}AI$ and I is non-singular, A is similar to A.

Hence similarity of matrices is reflexive.

Now, let $A,B \in S$ and let A be similar to B.

$\therefore A=P^{-1}BP$,where $P\in S$ is a non-singular matrix.

Now, $P^{-1}BP=A\rightarrow PP^{-1}BPP^{-1}=PAP^{-1}$

$\rightarrow B=PAP^{-1}$

$\rightarrow B=(P^{-1})^{-1}A(P^{-1})$.

Since P is non-singular $P^{-1} \in S$ is also non-singular.

$\therefore$B is similar to A.

Hence similarity to matrices is symmetric.

Now, let A,B,C $\in$ S.

Let A be similar to B , B be similar to C. Hence there exists non-singular matrices P,Q $\in$ S such that

$A=P^{-1}BP$ and $B=Q^{-1}CQ$.

Now, $A=P^{-1}BP$

$=P^{-1}(Q^{-1}CQ)P$

$=(P^{-1}Q^{-1})CQP$

$=(QP)^{-1}C(QP)$

Since P,Q $\in$ S are non-singular ,QP $\in$ S is also non-singular.

Hence A is similar to C.

$\therefore$ Similarity of matrices is transitive.

Hence similarity of matrices is an equivalence relation.

Problem 2 If A and B are similar matrices show that their determinants are same.

Solution

Let A and B be two similar matrices.

$\therefore$ There exists a non-singular matrix P such that $B=P^{-1}AP$.

Now, $|B|=|P^{-1}AP|$

$=|P^{-1}||A||P|$

$=|A|$ (since $|P^{-1}|=\frac{1}{|P|}$)

Hence the result.

Rank of a matrix

Definition:

Let $A=(a_{ij})$ be an $m\times n$ matrix. The rows Ri=(ai1,ai2,….,ain) of A can be thought of as elements of F^n. The subspace of F^n generated by the m rows of A is called the row space of A.

Similarly, the subspace the F^m generated by the n columns of A is called the column spaces of A. The dimension of the row space (column space) of A is called the row rank(column rank) of A.

Theorem:7.27

Any two row equivalent matrices have the same row space and have the same row rank .

Proof:

Let A be an $m\times n$ matrix .

It is enough if we prove that the row space of A is not altered by any elementary row operation.

Obviously the row space of A is not altered by an elementary row operation of the type R_i<->R_j.

Now,consider the elementary row operation R_i-> where c€F-{0}.

Since $L(\{R_1,R_2,\ldots,R_i,\ldots.,R_n\})=L(\{R_1,R_2,\ldots.,cR_i,\ldots.,R_n\})$ the row space of A is not altered by this type of elementary row operation .

Similarly we can easily prove that the row space ofA is not altered by an elementary row operation of the type R_i->$R_i + c\, R_i$.

Hence row equivalent matrices have the same row space and hence the same row rank.

Theorem:7.28

Any two column equivalent matrices have the same column rank.

Proof:

Let A be an $m\times n$ matrix.

It is enough if we prove that the column space of A is not altered by any elementary column operation.

Obviously the column space if A is not altered by an elementary column operation of the type C_i<->C_j.

Now, consider the elementary column operation C_i->sC_i where C €F-{0}.

Since $Lc\{C_1,C_2,\ldots.,C_i,\ldots,C_n\}=L(\{C_1,C_2,\ldots.,_{CCi},\ldots.,C_n\})$ the column space of A is not altered by this type of elementary column operation .

Similarly we can easily prove that the column space of A is not altered by an elementary column operation of the type C_i-> $C_i+ sC_i$

Hence column equivalent matrices have the same column space and hence the same column rank.

Theorem:7.29

The row rank and the column rank of any matrix are equal.

Proof:

Let $A=(a_{ij})$ be an m×n matrix.

Let $R_1,R_2,\ldots,R_m$ denote the rows of A.

Hence $R_i=(a_{i1},a_{i2},\ldots,a_{1n})$

Suppose the row rank of A is r.

Then the dimension of the row space is r.Let $V_1 = (b_{11},\ldots,b_{1n})$, $V_2 =(b_{21},\ldots,b_{2n})$, $V_r =(b_{r1},\ldots b_{rn})$ be a basis for the row space of A.

Then each row is a linear combination of the vectors $v_1,v_2,\ldots,v_r$

Let $R_1 =k_{11}v_1+k_2v_2+\ldots+k_{1r}v_r$

$R_2 =K_{21}V_1+K_{22}V_2+\ldots+K_2v_r$

……………. ………………………

$R_m =k_{m1}v_1,k_{m2}v_2+\ldots+k_{mr}v_r$

Where k_{ij} €F

Equating the ith component of each of the above equations we get

$a_{1i} =k_{11}b_{1i}+k_{12}b_{2i}+\ldots+k_{1r}b_{ri}$

$a_{2i} =k_{21}b_{1i}+k_{22}b_{2i}+\ldots+k_{2r}b_{ri}$

……………. …………………………

$a_{mi} =k_{m1}b_{1i}+k_{m2}b_{2i}+\ldots k_{mr}b_{ri}$

Hence
$$\begin{bmatrix} a_{1i} \\ . \\ . \\ A_{mi} \end{bmatrix} = b_{1i} \begin{bmatrix} k_{11} \\ . \\ . \\ . \\ K_{m1} \end{bmatrix} + \begin{bmatrix} k_{12} \\ . \\ . \\ . \\ k_{m2} \end{bmatrix} + \ldots + \begin{bmatrix} k_{1r} \\ . \\ . \\ k_{mr} \end{bmatrix}$$

Thus each column of A is a linear combination of r vectors.

Hence the dimension of the column space <= r.

Column rank of A<=r= Row rank of A.

Similarly , row rank of A<= Column rank of A

Hence the row rank and the column rank of A are equal.

Definition

The rank of a matrix A is the commion value of its row and column rank.

Note: 1

Since the row rank and the column rank of a matrix are unaltered by elementary row and column operation equivalent matrixs have the same rank.

In particular if a matrix a is reduces to canonical from $\begin{bmatrix} Ir & 0 \\ 0 & 0 \end{bmatrix}$ then rank of A=r

Thus to find the rank of a matrix A,we reduce A to the connonical and from find tha number of non-zero enties in the diagonal

Note that in the conenial form of the matrix A there exists an r×r sub-matrix namely ,$I_{r,}$ where determination is not zero.

Furthere every (r+1)×(r+1) sub –matrix contain a row of zeros and hence its determination is zero.

Also under any etementary row or column operation the value of a determination is either unaltened or multiplied by a non-zero constant

Hence then matrix A is also such that

i) There existe an rxr sub-matrix whose determinant is non zero .
ii) The determinant of every (r+1) x (r+1) sub-matrix is zero.

Hence one can also define the rank of a matrix. A to be rit A satisfices (i) and (ii).

Note: 2

Any non-singular matrix of order n is equivalent to the identity matrix and heme its rank is n.

Note: 3

The rank of a matrix is not altered on multiplication by non-singular matrices since premultiplication by a non-singular matrix is equivalent to applying elementary column operations.

Problem: 1

Find the rank of the matrix A = $\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 7 \end{bmatrix}$

Solution

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & 3 & 6 & 7 \\ 0 & 1 & 2 & 7 \end{bmatrix} \quad C_2 \leftrightarrow C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & -5 & -10 & -5 \\ 0 & 1 & 2 & 7 \end{bmatrix} \quad C_2 \rightarrow C_2 \sim$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & -10 & -5 \\ 0 & 1 & 2 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 1 & 0 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -5 & 0 & 0\\ 0 & 0 & 6 & 0\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$

∴ Rank of A = 3

Problem: 3

Find the rank of the matter A = $\begin{bmatrix}1 & 1 & 1 & 1\\ 4 & 1 & 0 & 2\\ 0 & 3 & 4 & 2\end{bmatrix}$ by examining the determination miners.

$$\begin{vmatrix}1 & 1 & 1\\ 4 & 1 & 0\\ 0 & 3 & 4\end{vmatrix} = 1\ [4-0] – [16-0] + 1\ [12-0]$$

= 16-16

=0

$$\begin{vmatrix}1 & 1 & 1\\ 1 & 0 & 2\\ 3 & 4 & 2\end{vmatrix} = 1[0-3]\ -1\ [2-6]\ +[4-0]$$

=-8 +8

=0

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 0 \\ 0 & 3 & 4 \end{vmatrix} = 0 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 4 & 2 \end{vmatrix}$$

$=-12 + 12$

$=0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 0 & 2 \\ 0 & 4 & 2 \end{vmatrix} = 1[0-8] -1\,[8-0] + [16 -0]$$

$=-16 + 16$

$=0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 2 \\ 0 & 3 & 2 \end{vmatrix} = 0 = 0 = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 0 & 2 \\ 0 & 4 & 2 \end{vmatrix}$$

Every 3x3 sub matrix of A has determinant Zero.

Also, $\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = 1 - 4 = -3 \neq 0$

$\therefore$ Rank of A $= 2$

Simultaneous Linear Equation

In this section we shall apply the theory of matrices developed in the preceeding sections to study the existence of solution of simultaneous linear equations

Matrix form of a set of linear equations

Consider a system of m linear equations in n unknowns $x_1, x_2, \ldots\ldots, x_n$ given by

$$a_{11}x_1 + a_{12}x_2 + \cdots \ldots + a_{1n}x_n = b_n$$

$$a_{21}x_1 + a_{22}x_2 + \cdots \ldots\ldots + a_{2n}x_n = b_2$$

. ...

....

$$a_{m1}x_1 + a_{m2}x_2 + \cdots \ldots\ldots + a_{mn}x_n = b_m$$

Using the concept of matrix multiplication and equality of matrices this system can be written as AX=B where,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ & \ldots\ldots\ldots & \\ & \ldots\ldots\ldots & \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{bmatrix}$$

The m x n matrix A is called the coefficient matrix.

Definition

A set of values of $x_1, x_2, \ldots\ldots . x_n$ which satisfy the above system of equations is called a solution of the system .The system of equations is said to be **consistent** if it has at least one solutions .Otherwise the system is said to be **inconsistent**.

The mx (n+1) matrix given by

$$A=\begin{bmatrix} a_{11} & \ldots & a_{1n} & b_1 \\ a_{21} & \ldots & a_{2n} & b_n \\ & \cdots\cdots & & \\ & \cdots\cdots\cdots & & \\ a_{m1} & & a_{mn} & b_m \end{bmatrix}$$

is called the **augmented matrix** of the system and denoted by (A,B).

Thus the augmented matrix (A, B) is obtained annexing to A the column matrix B, which become the (n+1) th column in (A, B) .

Note

Since every column in A appears in (A, B) the column space of the matrix A is a subspace of the column space of the matrix (A, B).

Hence the rank of A≤ rank of (A, B).

Theorem 7.30. The system of linear equations

AX = B is consistent iff rank of A = rank of (A, B).

Proof.

Let the system be consistent.

Let $u_1, \; u_2, \ldots\ldots u_n$ be a solution of the system

Then B $=u_1C_1+u_2C_2+\ldots\ldots+u_nC_n$ when $C_1, C_2, \ldots\ldots C_n$ denote the columns of A.

Hence then column space of the augmented matrix (A, B) namely $\langle C_1, C_2, \ldots\ldots, C_n\rangle$ is the same as the column space $\langle C_{,1}\; C_2, \ldots\ldots, C_n\rangle$ of A.

Hence the rank of A = rank of (A, B).

Conversely let rank of A = rank of $\langle A, B\rangle$.

Then the column rank of A = column rank of $\langle A, B\rangle$.

dim $\langle C_{1,}C_2, \ldots\ldots, C_n\rangle$ = dim $\langle C_1, C_2, \ldots\ldots, C_n B\rangle$.

But $\langle C_1, C_2, \ldots\ldots, C_n\rangle$ is a subspace ($C_1, C_2, \ldots\ldots, C_n$,B).

B is linear combination of $C_1, C_2, \ldots\ldots, C_n$

If B = $u_1C_{1+}u_2C_{2+} \dots \dots +. u_nC_n$ then $u_1, u_2, \dots \dots, u_n$ is solution of the system.

Hence the theorem.

Remark

The solution of a given system of simultaneous equations is not altered by interchanging any equations or by multiplying any equation by a non-constant or by adding a multiple of one equation another. Hence we can reduce the given system of equations to an equivalent system by applying elementary row operations to the augmented matrix. This redused form will enable us to test for the consistency and to find the solution if it exists. This is illustrated the following problems.

Solved problems

Problem-1

Show that the equations

x+y+z=6

x+2y+3z=14

x+4y+7z=30

consistent and solve them.

Solution

The given system of equations can be put the matrix form

$$AX=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 7\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}6\\ 14\\ 30\end{pmatrix}=B$$

The augmented matrix is given by

$$(A, B)=\begin{bmatrix}1 & 1 & 1 & 6\\ 1 & 2 & 3 & 14\\ 1 & 4 & 7 & 30\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & 1 & 2 & 8\\ 0 & 3 & 6 & 24\end{bmatrix} \quad \begin{matrix}R_2 \to & R_{2\,-} & R_1\\ R_3 \to & R_{3-} & R_1\end{matrix}$$

$$\sim\begin{bmatrix}1&1&1&6\\0&1&2&8\\0&0&0&0\end{bmatrix} R_3 \rightarrow R_{3-3}\mathrm{R}_2$$

Hence rank of A = rank of (A, B)=2

Also the given system of equation reduces to

$$\begin{bmatrix}1&1&1\\0&1&2\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\8\\0\end{bmatrix}$$

x+y+z=6

y+2z=8

Putting z =c we obtain the general solution of the system as x=c-2, y=8-2c, z=c.

Problem-2

Verify whether the following system of equations is consistent. If it is consistent, find the solution.

x-4y-3z=-16

4x-y+6z=16

2x+7y+12z =48

5x-5y+3z=0.

Solution

The matrix from of the system is given by

$$\begin{bmatrix}1&-4&-3\\4&-1&6\\2&7&12\\5&-5&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-16\\16\\48\\0\end{bmatrix}$$

The augmented matrix is given by

$$(A, B)=\begin{bmatrix}1&-4&-3&-16\\4&-1&6&16\\2&7&12&48\\5&-5&3&0\end{bmatrix}$$

$$\sim\begin{bmatrix}1&-4&-3&-16\\0&15&18&80\\0&15&18&80\\0&15&18&80\end{bmatrix}\begin{matrix}\mathrm{R}_2\rightarrow\mathrm{R}_{2-}4\mathrm{R}_1\\\mathrm{R}_3\rightarrow\mathrm{R}_{3-}2\mathrm{R}_1\\\mathrm{R}_4\rightarrow\mathrm{R}_{4-}5\mathrm{R}_1\end{matrix}$$

$$\sim \begin{bmatrix} 1 & -4 & -3 & -16 \\ 0 & 15 & 18 & 80 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \rightarrow R_{3-}R_2 \\ R_4 \rightarrow R_{4-}R_2 \end{matrix}$$

Rank of A = Rank of (A, B) = 2 and hence the system is consistent. Also the system of equations reduces to

$$\begin{pmatrix} 1 & -4 & -3 \\ 0 & 15 & 18 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -16 \\ 80 \\ 0 \\ 0 \end{bmatrix}$$

x-4y-3z=1 and 15y+18z=80.

Putting as x=$\left({}^{9c}/_{5}\right)$+$\left({}^{16}/_{3}\right)$,

y= - $\left({}^{6c}/_{5}\right)$+$\left({}^{16}/_{3}\right)$

z=c.

Problem-3

For what values of η the equations

x+y+z=1

x+2y+4z=η

x+4y+10z=$\eta^2$are consistent?

Solution

The matrix from of the system is given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \eta \\ \eta^2 \end{bmatrix}$$

The augmented matrix is given by

$$(A, B) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \eta \\ 1 & 4 & 10 & \eta^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \eta - 1 \\ 0 & 3 & 9 & \eta^2 - 1 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_{2-}R_1 \\ R_3 \rightarrow R_{3-}R_1 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \eta-1 \\ 0 & 0 & 0 & \eta^2-3\eta+2 \end{bmatrix} R_3 \rightarrow R_3-3R_2$$

The given system is consistent iff η^2-3η+2=0

η=2 or 1

Problem -4

Show that the system of equation

x+2y+z=11

4x+6y+5z=8

2x+2y+3z=19 is inconsistent.

Solution

The matrix form of the system is given by

$$\begin{bmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \\ 19 \end{bmatrix}$$

The augmented matrix is given by

$$(A, B) = \begin{bmatrix} 1 & 2 & 1 & 11 \\ 4 & 6 & 5 & 8 \\ 2 & 2 & 3 & 19 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 2 & 1 & 11 \\ 0 & -2 & 1 & -36 \\ 0 & -2 & 1 & -3 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2-4R_1 \\ R_3 \rightarrow R_3-2R_1 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 2 & 1 & 11 \\ 0 & -2 & 1 & -36 \\ 0 & 0 & 0 & 33 \end{bmatrix} R_3 \rightarrow R_3-R_2$$

Rank of A =2 and rank of (A, B) = 3

The given system is inconsistent.

7.7 Characteristic Equation And Cayley Hamilton Theorem

Definition:

An expression of the form $A_0 + A_1x + A_2x^2 + \ldots + A_nX^n$ where $A_0, A_1, \ldots A_n$ are square matrices of the same order and $A_n \neq \mathbf{0}$ is called the **matrics polynomial** of degree n.

For example,

$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} x + \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} x^2$ is a matrix polynomial of degree 2 and it is simply the matrix $\begin{pmatrix} 1+x+x^2 & 2+x \\ 2x+3x^2 & 3+x+x^2 \end{pmatrix}$.

Definition:

Let A be any square matrix of order n and let I be The identity matrix of order n. Then the matrix polynomial given by $A - xI$ is called the **Characteristic matrix** of A.

The determinant $|A - xI|$ which is an ordinary polynomial in x of degree n is called the **characteristic polynomial** of A.

The equation $|A - xI| = 0$ is called the **characteristic equation** of A.

Example 1:

Let A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Then the characteristic matrix of A is $A - xI$ given by

$$A - xI = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - x\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - x & 2 \\ 3 & 4 - x \end{pmatrix}$$

Therefore the characteristic polynomial of A is

$$|A - xI| = \begin{vmatrix} 1 - x & 2 \\ 3 & 4 - x \end{vmatrix}$$

$$= (1 - x)(4 - x) - 6$$

$$= x^2 - 5x - 2$$

Therefore the characteristic equation of A is $|A - xI| = 0$

Therefore $x^2 - 5x - 2 = 0$ is the characteristic equation of A.

Example 2:

Let A = $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

The characteristic matrix of A is $A - xI$ given by

$$A - xI = \begin{pmatrix} 1 - x & 0 & 2 \\ 0 & 1 - x & 2 \\ 1 & 2 & -x \end{pmatrix}$$

The characteristic polynomial of A is

$$|A - x\,I| = \begin{vmatrix} 1-x & 0 & 2 \\ 0 & 1-x & 2 \\ 1 & 2 & -x \end{vmatrix}$$

$$= (1-x)[(1-x)(-x) - 4] - 2(1-x)$$

$$= -x(1-x)^2 - 4(1-x) - 2 + 2x$$

$$= -x^3 + 2x^2 - x - 4 + 4x - 2 + 2x$$

$$= -x^3 + 2x^2 + 5x^2 - 6$$

The characteristic equation of A is

$$-x^3 + 2x^2 + 5x^2 - 6 = 0$$

(i.e) $x^3 - 2x^2 - 5x^2 + 6 = 0$

Theorem 7.31: (Cayley Hamilton Theorem)

Any square matrix A satisfies its characteristic equation.

(i.e) if $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is the characteristic polynomial of degree n of A then

$$a_0I + a_1A + a_2A^2 + \ldots + a_nA^n = \mathbf{0}.$$

Proof:

Let A be a square matrix of order n.

Let $|A - x\,I| = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ ……………………………..(1)be the characteristic polynomial of A.

Now, adj$(A - x\,I)$ is a matrix polynomial of degree $n - 1$ since each entry of the matrix adj$(A - x\,I)$ is a cofactor of $A - x\,I$ and hence is a polynomial of degree $\leq n - 1$.

Therefore let adj(A – x I) = $B_0 + B_1x + B_2x^2 + . . . + B_{n-1}x^{n-1}$ ………...(2) Now,

$(A - xI)\mathrm{adj}(A - xI) = |A - xI|\, I$ (since (adj A)A = A (adj A) = $|A|\, I$)

Therefore $(A - xI)(B_0 + B_1x + B_2x^2 + . . . + B_{n-1}x^{n-1})$

$$= (a_0 + a_1x + a_2x^2 + . . . + a_nx^n)\, I \text{ using (1) and(2)}$$

Therefore Equating the coefficients of the corresponding powers of x we get

$$AB_0 = a_0 I$$

$$AB_1 - B_0 = a_1 I$$

$$AB_2 - B_1 = a_2 I$$

……….. ……

……….. ……

$$AB_{n-1} - B_{n-2} = a_{n-1} I$$

$$- B_{n-1} = a_n I$$

Pre-multiplying the above equations I, A, A^2, ….., A^n respectively and adding we get

$$a_0 I + a_1A + a_2A^2 + . . . + a_nA^n = \mathbf{0}.$$

<u>Note:</u>

The inverse of a non-singlar matrix can be calculated by using the Cayley Hamilton theorem as follows.

Let $a_0 + a_1x + a_2x^2 + . . . + a_nx^n$ be the characteristic polynomial of A.

Then by theorem 1.1, we have

$$a_0 I + a_1A + a_2A^2 + . . . + a_nA^n = \mathbf{0} \qquad \text{…………………………… (3)}$$

Since $|A - xI| = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ we get $a_0 = |A|$ (by putting $x = 0$)

Therefore $a_0 \neq 0$ (since A is a non singular matrix)

Therefore $I = -\frac{1}{a_0}[a_1A + a_2A^2 + \ldots + a_nA^n]$ (by 3)

$$A^{-1} = -\frac{1}{a_0}[a_1I + a_2A + \ldots + a_nA^{n-1}]$$

Problem:1

Find the characteristic equation of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{bmatrix} \mathbf{3}$$

Solution:

The characteristic equation of A is given by $|A-XI| = 0$.

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{vmatrix} 8-x & -6 & 2 \\ -6 & 7-x & -4 \\ 2 & -4 & 3-x \end{vmatrix} = 0$$

(8-x) [(7-x)(3-x)-16] + 6[-6(3-x)+8] + 2(24-2(7-x)] =0

(8-x) [21-7x-3x+x^2-16] + 6 [-18+6x+8] +2 (24-14+2x) = 0

(8-x) (x^2-10x+15) + 6 (6x-10)+2 (2x+10) = 0

(8x^2-80x+40-x^3+ (10x^2-5x)+ (36-60) + (4x+20) = 0

X^3-18x^2+45x = 0.

Which represents the characteristic equation of A.

Problem:2

Show that the non-singular matrix A = $\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{1} \end{pmatrix}$ satisfies the equation A^2-2A-5I = 0. Hence evaluate A^{-1}.

Solution:

The characteristic polynomial of A is

$$|A-XI| = \begin{vmatrix} 1-x & 2 \\ 3 & 1-x \end{vmatrix}$$

= 3(1-x) (1-x)-6

= 1-x+x+x^2-6

= x^2-2x-5

By cayley Hamilton theorem A^2-2A-5I = 0

-5I = $-A^2+2A$

I = 1/5 (A^2-2A)

$\therefore A^{-1}$ = 1/5 (A-2I)

$$= 1/5 \left[\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

$$= 1/5 \begin{bmatrix} 1-2 & 2-0 \\ 3-0 & 1-2 \end{bmatrix}$$

$$= 1/5 \begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}$$

Problem:3

Show that the matrix A = $\begin{bmatrix} \mathbf{2} & \mathbf{-3} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{-3} \\ \mathbf{-5} & \mathbf{2} & \mathbf{-4} \end{bmatrix}$ **satisfies the equation on A (A-I) (A+2I) = 0**

Solution:

The characteristic polynomial of A is

$$|A-XI| = \begin{vmatrix} 2-x & -3 & 1 \\ 3 & 1-x & 3 \\ -5 & 2 & -4-x \end{vmatrix}$$

= (2-x)[(1-x)(4-x)-6] + 3[3(4-x)+5]+ [6-(1-x) (-5)]

= (2-x) [-4x+4x+x^2-] + 3[[-12-3x+15] + [6+5-5x]

= (2-x) [x^2+ 3x-10] + 3[-3x+3] + [-5x + 1]

= $2x^2$+ 6x-20-x^3-$3x^2$+ 10x-9x+ 9-5x+11

= $-x^3-x^2+2x$

$\therefore$ By cayley Hamilton theorem $A^3-A^2+2A = 0$

$A^3+A^2-2A = 0$

Hence A (A^2+A-2I) = 0

A (A+ 2I) (A-I) =0

Problem:4

Using Cayley Hamilton theorem find the inverse of the matrix $\begin{bmatrix} \mathbf{7} & \mathbf{2} & \mathbf{-2} \\ \mathbf{-6} & \mathbf{-1} & \mathbf{2} \\ \mathbf{6} & \mathbf{2} & \mathbf{-1} \end{bmatrix}$

Solution:

Let A = $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic polynomial of A = $|A-xI|$

$$= \begin{vmatrix} 7-x & 2 & -2 \\ -6 & -1-x & 2 \\ 6 & 2 & -1-x \end{vmatrix}$$

= (7-x) [(1-x)(-1-x)-4] -2 [-6(1-x)-12] -2 [-12-6(1-x)]

= (7-x) [1+x+x+x^2-4] -2 [6+6x-12] -2 [-12+6+6x]

= (7-x) (x^2+2x-3)-12(x-1)-12(x-1)

= 7x^2+ 14x-21-x^3+2x^2+3x-12x+12-12x+12

= -x^3+5x^2-7x+3

∴ By cayley Hamilton theorem,

$-A^3+5A^2-7A+3I_3 = 0$

$A^3-5A^2+7A+3I_3 = 420$

$3I_3 = A^3-5A^2+7A$

$I_3 = 1/3\ (A^3-5A^2+7A)$

Pre multiplying by A^{-1}on oth sides we get

$A^{-1} = 1/3\ [A^3-5A^2+7A]$ ………. (1)

Now,

$$A^2 = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 49-12-12 & 14-2-4 & -14+4+2 \\ -42+6-12 & -12+1+4 & 12-2+2 \\ 42-12-6 & 12-2-2 & -12+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$$

From (1)

$$A^{-1} = 1/3 \left[\begin{pmatrix} 25 & 8 & -8 \\ -24 & 7 & 8 \\ 24 & 8 & -7 \end{pmatrix} - \begin{pmatrix} 35 & 10 & -10 \\ -30 & -5 & 10 \\ 30 & 10 & -5 \end{pmatrix} + \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix} \right]$$

$$=1/3 \begin{bmatrix} 25-35+7 & 8-10-0 & -8+10+0 \\ -24+30+0 & -7+5+7 & 8-10+0 \\ 24-30+0 & 8-10+0 & -7+5+7 \end{bmatrix}$$

$$= 1/3 \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2/3 & 2/3 \\ 2 & 5/3 & -2/3 \\ -2 & -2/3 & 5/3 \end{bmatrix}$$

<u>Problem:5</u>

Find the inverse of the matrix

$$\begin{bmatrix} \mathbf{3} & \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{-3} & \mathbf{4} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix} \textbf{ using cayely hamilton theorem}$$

<u>Solution:</u>

The characteristic polynomial of A = $|A-XI|$

$$= \begin{vmatrix} 3-x & 3 & 4 \\ 2 & -3-x & 4 \\ 0 & -1 & 1-x \end{vmatrix}$$

= (3-x) [(3-x) (1-x) +4] – 3 [2(1-x) -0] +4[2-0]

= (3-x) [$-3x+3x+x^2+4$] -3 [2-2x]-8

= (3-x)[x^2+2x+1] -6+6x-8

$= 3x^2+6x+3-x^3-2x^2-x+6x-14$

$= -x^3+x^2+11x-11$

$\therefore$ By cayely Hamilton theorem

$-A^3+A^2+11A-11I_3= 0$

$\therefore A^3-A^2-11A+11I_3 = 0$

$11I_3 = -(A^3-A^2-11A)$

$I_3 = -1/11 [A^3-A^2-11A]$

Pre multiplying by A^{-1} on both sides we get,

$A^{-1} = -1/11 [A^2-A-11I_3]$ ………(!)

$$A^2 = \begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9+6+0 & 9-9-4 & 12+12+4 \\ 6-6+0 & 6+9-4 & 8-12+4 \\ 0-2+0 & 0+3-1 & 0—4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -4 & 28 \\ 0 & 11 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

From (1)

$$A^{-1} = -1/11 \left[\begin{pmatrix} 15 & -4 & 28 \\ 0 & 11 & 0 \\ -2 & 2 & -3 \end{pmatrix} \begin{pmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix} \right]$$

$$= -1/11 \begin{bmatrix} 15-3-11 & -4-3-0 & 28-4-0 \\ 0-2-0 & 11+3-11 & 0-4-0 \\ -2-0-0 & 2+1-0 & -3-1-11 \end{bmatrix}$$

$$= \begin{bmatrix} -1/11 & 7/11 & -211/11 \\ 2/11 & -3/11 & 4/11 \\ 2/11 & -3/11 & 15/11 \end{bmatrix}$$

Problem:6

Verify cayley Hamilton theorem for the matrix A = $\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{4} & \mathbf{3} \end{pmatrix}$

Solution:

The characteristic of equation is $|A-XI| = 0$

$\therefore \begin{vmatrix} 1-x & 2 \\ 4 & 3-x \end{vmatrix} = 0$

(1-x) (3-x) -8 = 0

$3-x-3x+x^2-8 = 0$

$X^2-4x-5 = 0$

By Cayley Hamilton theorem A satisfies is characteristic equation

$\therefore$ we have $A^2-4A-5I = 0$

$A^2 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 4 \end{pmatrix}$

$= \begin{bmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} & \begin{pmatrix} 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{bmatrix}$

$= \begin{bmatrix} 1+8 & 2+6 \\ 4+12 & 8+9 \end{bmatrix}$

$= \begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix}$

$A^2-4A-5I = \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix} - \begin{pmatrix} 4 & 8 \\ 16 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

$= \begin{pmatrix} 9-4-5 & 8-8-0 \\ 16-16-0 & 7-12-5 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$= 0$

$\therefore$ Theorem is verified.

Problem:7

Using cayley Hamilton theorem for the matrix A = $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-2} \\ \mathbf{2} & \mathbf{2} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} \end{bmatrix}$ **find (i)** $\mathbf{A^{-1}}$ **(ii)** $\mathbf{A^{4}}$

Solution:

The characteristic equation of A is $|A-XI| = 0$

$$\begin{vmatrix} 1-x & 0 & 2 \\ 2 & 2-x & 4 \\ 0 & 0 & 2-x \end{vmatrix} = 0$$

(1-x) [(2-x) (2-x)-0] – 0[-2(0-0)] = 0

$(1-x)\,(4-2x-2x+x^2) = 0$

$(1-x)\,(x^2-4x+4) = 0$

$X^2-4x+4-x^3+4x^2-4x = 0$

$X^3+5x^2-8x+4 = 0$

$X^3-5x^2+8x- 4 = 0$

By cayley Hamilton theorem

$A^3- 5A^2+8A-4I = 0$ ………(1)

$4I = A^3-5A^2+8A$

$I = ¼[A^3-5A^2+8A]$

i) To find A^{-1} pre multiplying by A^{-1} we get

$A^{-1}= ¼\,[A^2-5A+8I]$ ………(2)

$$A^2 = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{bmatrix} 1+0-0 & 0+0+0 & -2+0-4 \\ 2+4+0 & 0+4+0 & -4+8-8 \\ 0+0+0 & 0+0+0 & 0+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix}$$

From (2)

$$A^{-1} = ¼\left[\begin{pmatrix}1 & 0 & -6\\ 6 & 4 & 12\\ 0 & 0 & 4\end{pmatrix} - \begin{pmatrix}5 & 0 & -10\\ 10 & 10 & 20\\ 0 & 0 & 10\end{pmatrix} + \begin{pmatrix}8 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 8\end{pmatrix}\right]$$

$$= ¼\begin{bmatrix}1-5+8 & 0-0+0 & -6+10+0\\ 6-10+0 & 4-10+8 & 12-20+0\\ 0-0+0 & 0-0+0 & 4-10+8\end{bmatrix}$$

$$= ¼\begin{bmatrix}4 & 0 & 4\\ -4 & 2 & -8\\ 0 & 0 & 2\end{bmatrix}$$

$$= \begin{pmatrix}1 & 0 & 1\\ -1 & 1/2 & -2\\ 0 & 0 & 1/2\end{pmatrix}$$

ii) To find A^4

From (1)

$A^3 = 5A^2-8A+4I$

$A^4 = 5A^3-8A^2+4A$

$= 5\,[5A^2-8A+4I] - 8A^2 + 4A$

$= 25A^2-40A+20I-8A^2+4A$

$= 17A^2-36A+20A$

$$= 17\begin{pmatrix}1 & 0 & -6\\ 6 & 4 & 12\\ 0 & 0 & 4\end{pmatrix} - \begin{pmatrix}1 & 0 & -2\\ 2 & 2 & 4\\ 0 & 0 & 2\end{pmatrix} + \begin{pmatrix}20 & 0 & 0\\ 0 & 20 & 0\\ 0 & 0 & 20\end{pmatrix}$$

$$= \begin{pmatrix}17 & 0 & -102\\ 102 & 68 & 204\\ 0 & 0 & 68\end{pmatrix} - \begin{pmatrix}36 & 0 & -72\\ 72 & 72 & 144\\ 0 & 0 & 72\end{pmatrix} + \begin{pmatrix}20 & 0 & 0\\ 0 & 20 & 0\\ 0 & 0 & 20\end{pmatrix}$$

$$= \begin{bmatrix}17-36+20 & 0-0+0 & -102+72+0\\ 102-72+0 & 68-72+20 & 204-144+0\\ 0-0+0 & 0-0+6 & 68-72+20\end{bmatrix}$$

$$A^4 = \begin{bmatrix}1 & 0 & -30\\ 30 & 16 & 60\\ 0 & 0 & 16\end{bmatrix}$$